Lecture 7. Absorption and Material Dispersion

- Complex refractive index
- Material dispersion
- Classical theory - Lorentz oscillator model
- Signal velocity in a dispersive medium

Reading: Photonic Devices, Jia-Ming Liu, Chapter 1
Introduction to Modern Optics, G. R. Fowles, Chapter 6
Fundamentals of Photonics, 2nd edition, Saleh & Teich, Chapter 5 (5.5-5.6)

General remarks

- The study of the propagation of light through matter, particularly solid matter, comprises one of the important and interesting branches of optics.
- The many and varied optical phenomena exhibited by solids include *selective absorption, dispersion*, *double refraction*, *polarization effects*, *electro-optical effect*, *magneto-optical effect*, *acousto-optical effect* and *nonlinear optical effect*.
- Many of the optical properties of solids can be understood on the basis of *classical electromagnetic theory*.
- In this lecture we will discuss the application of Maxwell equations to the propagation of light through solids.
- We will treat the *microscopic* origin of the optical properties of solids classically. The classical theory gives considerable physical insight.

Medium with loss or gain

- ε becomes *complex* when a medium has an optical *loss* or *gain*.

$$k^2 = \omega^2\mu_o\varepsilon = \omega^2\mu_o\,(\varepsilon' + i\varepsilon'')$$

- The propagation constant k becomes *complex*

$$k = k' + ik'' = k' + i\alpha/2$$

- The index of refraction also becomes *complex*

$$n = [(\varepsilon' + i\varepsilon'')/\varepsilon_o]^{1/2} = n' + in''$$

- The relation between k and n is still valid: $k = 2\pi n'/\lambda_o + i2\pi n''/\lambda_0$
- The relation between α and imaginary n'': $\alpha = 2k'' = 4\pi n''/\lambda_0$

- The impedance $Z = Z_o/n$ of the medium also becomes *complex*. Therefore, ***E*** and ***H*** are *no longer in phase*. (recall $Z_0 = 377\ \Omega$)

- If we consider an optical wave propagating in the **z** direction, then a monochromatic plane wave complex electric field has the form (using $k = k' + i\alpha/2$)

$$\mathbf{E}(\mathbf{r}, t) = E_o \exp(-\alpha z/2) \exp i(k'z - \omega t)$$

⇒The amplitude is *not* constant but varies *exponentially* with z. Thus, light intensity (or *power*) varies exponentially with z:

$$I \propto \exp(-\alpha z)$$

Relating α and γ:

$$-\alpha = (1/L) \ln (I_o/I_i) = (1/L) \log(I_o/I_i)/\log(e) = -1/(10 \log(e))\ \gamma = -0.23\ \gamma$$

Attenuation/Gain Coefficients

- If $\alpha > 0$, as the optical wave propagates, its field amplitude and intensity decay exponentially along the direction of propagation.

Therefore, α is called the ***attenuation coefficient***.

- If $\alpha < 0$, the field amplitude and intensity of the optical wave grow exponentially. Then, we define $g = \alpha$ as the ***gain coefficient*** or *amplification coefficient*.

The unit of both α and g is per length (m^{-1}, cm^{-1}, mm^{-1}, μm^{-1}).

e.g. The complex dielectric constant ($\varepsilon/\varepsilon_o$) of semiconductor GaAs at an optical wavelength of λ_0 = 850 nm is 13.17 + i0.49. Therefore, at this wavelength, GaAs has a complex refractive index of

$$n = (\varepsilon/\varepsilon_o)^{1/2} = (13.17 + i0.49)^{1/2} = 3.63 + i0.0676$$

And an attenuation (absorption) coefficient of

$$\alpha = 2k'' = 4\pi n''/\lambda_0 = 1\ \mu m^{-1}$$

i.e. An optical beam at 850 nm wavelength can travel in GaAs only for a distance of $l = -\ln(0.01)/\alpha = 4.6\ \mu m$ before losing 99% of its energy to absorption, which is obtained by solving $\exp(-\alpha l) = 0.01$ with $\alpha = 1\ \mu m^{-1}$ [*Note: we will discuss semiconductor absorption later*.]

Material dispersion

- The index of refraction is in general ***frequency or wavelength dependent***. This is true for *all* transparent optical media.

- The variation of the index of refraction with frequency is called ***dispersion***. The dispersion of glass is responsible for the familiar splitting of light into its component colors by a prism.

- In order to explain the dispersion it is necessary to take into account the actual motion of the electrons in the optical medium through which the light is traveling. Here, we will discuss the theory of dispersion in detail.

Phase velocity in dielectric media

$$v_p = 1/\sqrt{(\mu_0\varepsilon)} = 1/\sqrt{(\mu_0\varepsilon_0\varepsilon_r)}$$

- The velocity of light in a dielectric medium is therefore

$$v_p = c/\sqrt{\varepsilon_r}$$

where we used the relation $\mu_0\varepsilon_0 = 1/c^2$ and c is the speed of light.

$$v_p = c/n$$

$$n = \sqrt{\varepsilon_r}$$

**The refractive index n is rooted in the material relative permittivity.*

Material dispersion

- *D*ispersion in the susceptibility of a medium is caused by the fact that the *response of the medium to excitation by an optical field does not decay instantaneously*.
- The general characteristics of the medium can be understood from its *impulse response*. *For optics, $\chi(t)$*
- In general, the impulse response of a medium decays *exponentially* while *oscillating* at some resonance frequencies. *Like a damped harmonic oscillator*
- There may exist several *exponential relaxation constants* and several *oscillation frequencies* for a given material across the electromagnetic spectrum. This is true even within the optical spectral region.
- At a given optical frequency ω, the characteristics of the material response are dominated by the resonance frequency ω_0 closest to ω and the relaxation constant associated with the oscillation at this particular resonance frequency.

Susceptibility in the time domain

- For simplicity, we consider a medium of a single *resonance frequency* at ω_0 with a *relaxation constant* γ.
- The susceptibility in the time domain $\chi(t)$ is the *impulse response* of the medium, which is *real* and has the following general form:

$$\chi(t) \propto \begin{matrix} e^{-\gamma t}\sin\omega_0 t, t > 0 \\ 0, t < 0 \end{matrix}$$

- Note that $\chi(t) = 0$ for $t < 0$ because a medium can respond only after, *but not before*, an excitation. This is the *causality* condition, which applies to *all* physical systems.

Susceptibility in the frequency domain

- The Fourier transform of the impulse response

$$\chi(\omega) = \int_{-\infty}^{\infty} \chi(t) e^{i\omega t} dt \approx -\chi_b \frac{\omega_0}{\omega - \omega_0 + i\gamma}$$

where $\chi_b = \chi(\omega << \omega_0)$ is a constant equal to the *background* value of $\chi(\omega)$ at *low* frequencies *far away* from resonance.

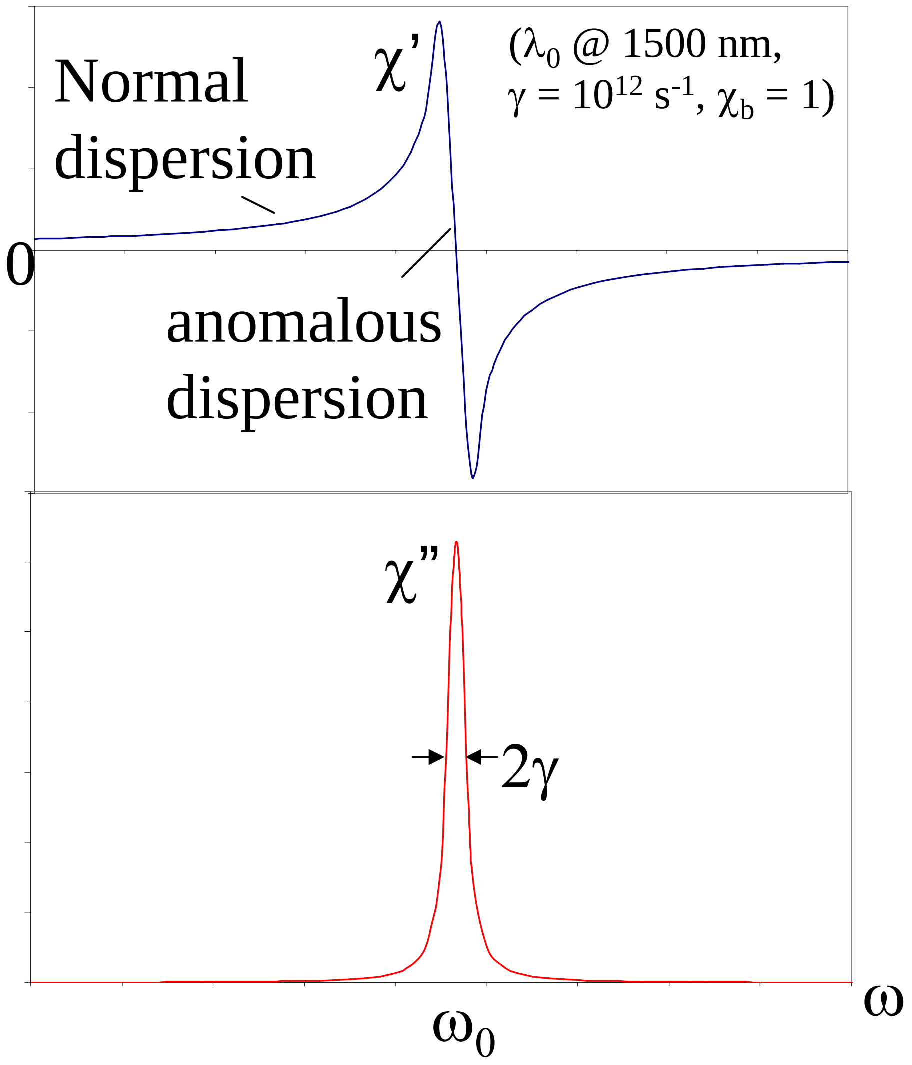
- Here we have dropped a term that contains $\omega+\omega_0$ in its denominator because $\omega+\omega_0 >> |\omega-\omega_0|$ in the optical spectral region. (so-called *rotating-wave approximation*)

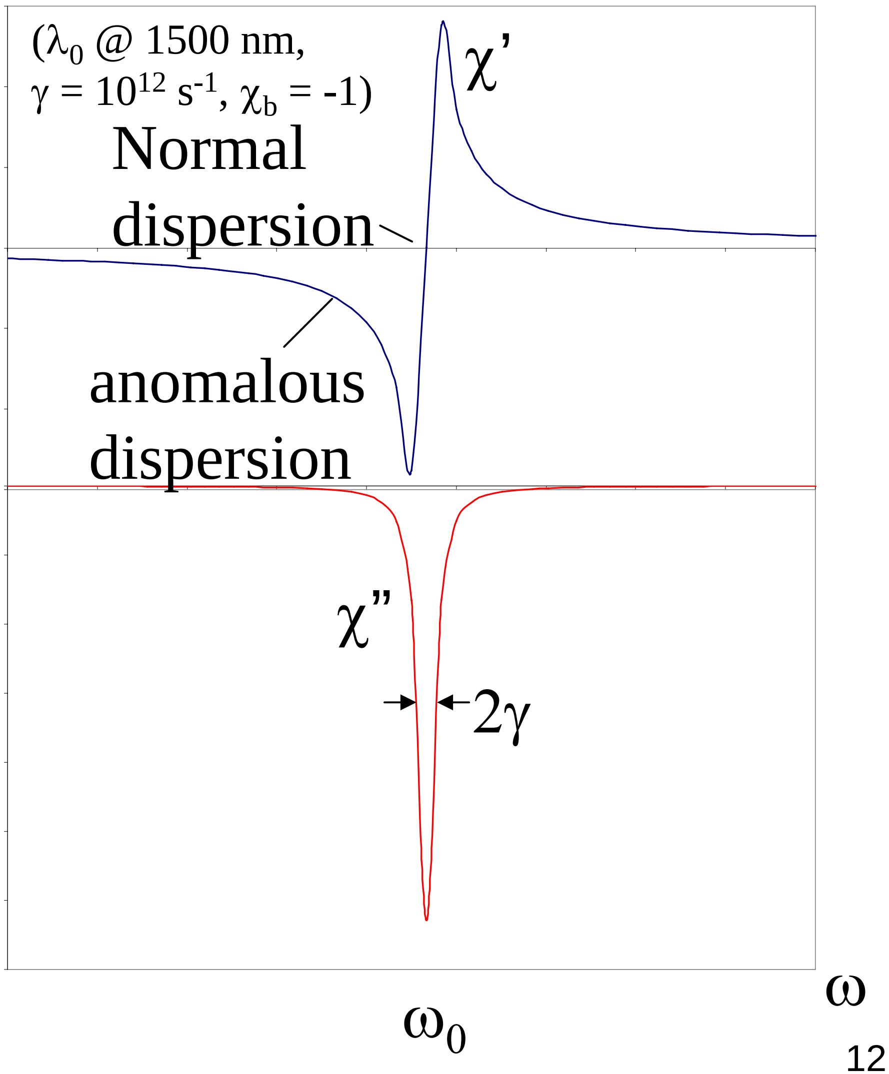
- This susceptibility has the following *real* and *imaginary* parts:

$$\chi'(\omega) = -\chi_b \frac{\omega_0(\omega - \omega_0)}{(\omega - \omega_0)^2 + \gamma^2} \qquad \chi''(\omega) = \chi_b \frac{\omega_0 \gamma}{(\omega - \omega_0)^2 + \gamma^2}$$

Real and imaginary parts of susceptibility

- For a medium with loss
- For a medium with gain





Susceptibility in the frequency domain

- Note that $\chi''(\omega)$ has a ***Lorentzian lineshape***, which has a FWHM $\Delta\omega = 2\gamma$. The sign of χ'' depends on that of χ_b.
- In the *normal* state, $\chi_b > 0$, and the medium has optical *loss* near resonance. This characteristic results in the *absorption* of light at frequency in the vicinity of $\omega = \omega_0$.
- When $\chi_b < 0$, the medium has optical *gain*, resulting in the *amplification* of light in the vicinity of $\omega = \omega_0$ such as in the case of a laser.
- Note that both χ' and χ'' are proportional to χ_b.
- Therefore, when χ'' changes sign, χ' also changes sign.
- When $\chi'' < 0$, χ' is negative for $\omega<\omega_0$ and positive for $\omega>\omega_0$.

Important material dispersion characteristics

- $\chi'(\omega \ll \omega_0)$ is larger than $\chi'(\omega \gg \omega_0)$ in the *normal* state. Therefore, around any single resonance frequency, ε' at any frequency on the low-frequency side has a value larger than that at any frequency on the high-frequency side.

- A medium is said to have ***normal dispersion*** in a spectral region where ε' increases with frequency s.t. $d\varepsilon'/d\omega > 0$. It is said to have ***anomalous dispersion*** in a spectral region where ε' decreases with increasing frequency s.t. $d\varepsilon'/d\omega < 0$.

  Because $dn/d\omega$ and $d\varepsilon'/d\omega$ have the same sign, the index of refraction also increases with frequency in a spectral region of *normal dispersion* and decreases with frequency in a spectral region of *anomalous dispersion*.

Important material dispersion characteristics

- When a material is in its *normal* state, normal dispersion appears everywhere except in the immediate neighborhood within the FHWM of a resonance frequency where anomalous dispersion occurs. This characteristic can be reversed near a resonance frequency where *resonant amplification*, rather than absorption, exists.

- Note the distinction between the definition of normal and anomalous dispersion in terms of the sign of $d\varepsilon'/d\omega$ or $dn/d\omega$ and that of positive and negative group-velocity dispersion in terms of the sign of D or D_λ. Both positive and negative group-velocity dispersion can appear in a spectral region where the dispersion defined in terms of $dn/d\omega$ is normal.
- In most transparent materials, such as glass and water, *normal dispersion* appears in the visible spectral region and may extend to the near infrared and near ultraviolet regions.

Observing anomalous dispersion

- Anomalous dispersion can be observed experimentally *if the substance is not too opaque at the resonance frequency*.

- E.g. certain dyes have absorption bands in the visible region of the spectrum and exhibit anomalous dispersion in the region of these bands.

- Prisms doped with these dyes produce a spectrum that is *reversed*, i.e. the *longer* wavelengths are *refracted more* than the shorter wavelengths.

Kramers-Kronig Relations

- A *dispersive* material, i.e. a material whose refractive index is wavelength dependent, *must be absorptive* and must exhibit an absorption coefficient that is also wavelength dependent.
- The relation between the *absorption coefficient* and the *refractive index* is a result of the ***Kramers-Kronig relations***, which relate the real and imaginary parts of the susceptibility of a medium, $\chi'(\upsilon)$ and $\chi''(\upsilon)$

$$\chi'(\nu) = \frac{2}{\pi}\int_0^\infty \frac{s\chi''(s)}{s^2-\nu^2}ds \qquad \chi''(\nu) = \frac{2}{\pi}\int_0^\infty \frac{\nu\chi'(s)}{\nu^2-s^2}ds$$

- Given the real or the imaginary component of $\chi(\upsilon)$ for *all* υ, Kramers-Kronig relations allow the complementary component to be determined for *all* υ. The Kramers-Kronig relations connecting $\chi''(\upsilon)$ and $\chi'(\upsilon)$ translate into relations between the *absorption coefficient* $\alpha(\upsilon)$ and the *refractive index* $n(\upsilon)$.
- The Kramers-Kronig relations are a special Hilbert-transform pair, as can be understood from *linear systems theory*. They are applicable for all *linear*, *shift-invariant*, *casual* systems with *real impulse response* functions.

Example on an atomic resonance

- For an atomic transition associated with absorption or emission of optical radiation at 1μm wavelength, the resonance frequency is

$$\upsilon_0 = c/\lambda = 300 \text{ THz}, \ \omega_0 = 2\pi\upsilon_0 = 1.885 \times 10^{15} \text{ s}^{-1}$$

- If the polarization associated with this resonant transition relaxes with a time constant of τ = 1 ps, then

$$\gamma = 1/\tau = 10^{12} \text{ s}^{-1}$$

$$\text{And } \Delta\omega = 2\gamma = 2 \times 10^{12} \text{ s}^{-2}.$$

- Thus the Lorentzian spectral line has a FWHM linewidth of

  $\Delta\upsilon = \Delta\omega/2\pi \approx$ 318 GHz (*which is quite broad but is approximately only 0.1% of the center frequency* υ_o).

- If the polarization relaxation time constant is τ = 1 ns, we find a spectral linewidth of $\Delta\upsilon \approx$ 318 MHz.

Classical theory

Lorentz oscillator model

Macroscopic fields and Maxwell equations

- *Recall* that the electromagnetic state of matter at a given point is described by *four* quantities:
  - The *volume density of free electric charge* ρ
  - The *volume density of electric dipoles*, called the *polarization* **P**
  - The *volume density of magnetic dipoles*, called the *magnetization* **M**
  - The *free electric current per unit area*, called the *current density* **J**.
- All of these quantities are considered to be *macroscopic* averaged in order to smooth out the microscopic variations due to the atomic makeup of all matter.
- They are related to the macroscopically averaged fields **E** and **H** by the following Maxwell equations:

Maxwell equations for nonmagnetic media with no free charge

- In most photonic applications (*except for magneto-optics*), we only consider *nonmagnetic* media that are *electrically neutral*.
- Hence **M** and ρ are both zero.

$$\nabla \times E = -\mu_0 \frac{\partial H}{\partial t}$$

$$\nabla \times H = \varepsilon_0 \frac{\partial E}{\partial t} + \frac{\partial P}{\partial t} + J$$

$$\nabla \bullet E = -\frac{1}{\varepsilon_0} \nabla \bullet P$$

$$\nabla \bullet H = 0$$

The general wave equation

- The general wave equation for the **E** field is obtained by taking the curl of the ∇×**E** equation and the time derivative of the ∇×**H** equation.

$$\nabla \times (\nabla \times E) + \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = -\mu_0 \frac{\partial^2 P}{\partial t^2} - \mu_0 \frac{\partial J}{\partial t}$$

- The two terms on the right-hand side are called source terms. They stem from the presence of ***polarization charges*** and ***conduction charges*** within the medium.

- The way in which the propagation of light is affected by the sources is revealed by the solution of the wave equation when the source terms are included.

Nonconducting media, metals, semiconductors

- In the case of ***nonconducting media*** the ***polarization term*** is of importance. It turns out that this term leads to an explanation of many optical effects including *dispersion*, *absorption*, *double refraction*, and *optical activity*, etc.

- In the case of ***metals*** it is the ***conduction term*** that is important and the resulting solutions of the wave equation explain the *large opacity* and *high reflectance* of metals.

- In the case of ***semiconductors both source terms*** *must be taken into account*. The result is a rather complicated wave equation and the solutions are somewhat difficult to interpret. Nevertheless, a qualitative description of many of the optical properties of semiconductors is furnished by classical theory.

Polarization in a nonconducting isotropic medium

- In a *nonconducting*, *isotropic* medium, the electrons are permanently bound to the atoms comprising the medium and there is no preferential direction.
- This is what is meant by a simple isotropic dielectric such as glass.
- Suppose that each electron, of charge –e, in a dielectric is displaced a distance **r** from its equilibrium position.
- The resulting macroscopic polarization **P** of the medium is given by

$$P = -Ner$$

where N is the number of electrons per unit volume.

Static polarization

- If the displacement of the electron is the result of the application of a *static* electric field **E**, and if the electron is elastically bound to its equilibrium position with a *force constant* K, then the force equation is

$$-eE = Kr$$

- The *static polarization* is therefore given by

$$P = \frac{Ne^2}{K} E$$

- However, *if the applied field* ***E*** *varies with time, the above equation is invalid*!

Lorentz oscillator model

- In order to find the true polarization **P**(t) in the *time-varying* field **E**(t), we must take the actual motion of the electrons into account.
- We consider the bound electrons as *classical damped harmonic oscillators*. The approach is known as the ***Lorentz oscillator model***.
- The *linear second-order ordinary differential equation* of motion is

$$m\frac{d^2r}{dt^2} + \underbrace{m\Gamma\frac{dr}{dt}}_{\text{Frictional damping force that is proportional to the velocity of the electron}} + Kr = -eE$$

Γ: damping coefficient (s^{-1})

- Note that the magnetic force e**v**×**B** is neglected. For electromagnetic waves, this force is normally smaller than the electric force e**E**.

Lorentz oscillator model

- Consider that the applied electric field **E**(t) varies harmonically with time according to exp(-iωt).
- Assuming that the motion of the electron has the same harmonic time dependence, we find that the differential equation of motion becomes

$$(-m\omega^2 - i\omega m\Gamma + K)r = -eE$$

- Therefore, the polarization **P**(t) = -Ne**r**(t) is given by

$$P = \frac{Ne^2}{-m\omega^2 - i\omega m\Gamma + K}E$$

- It reduces to the static value when ω = 0.

Frequency-dependent polarization

- Thus for a given amplitude of the applied electric field, the amount of polarization varies with frequency **P**(ω).
- The phase of **P**(t), relative to that of the electric field **E**(t), also depends on the frequency. This is shown by the presence of the *imaginary* term in the denominator.
- A more significant way of writing the polarization **P**(t) is

$$P = \frac{Ne^2/m}{\omega_0{}^2 - \omega^2 - i\omega\Gamma} E = \varepsilon_0 \frac{\omega_0{}^2 \chi_0}{\omega_0{}^2 - \omega^2 - i\omega\Gamma} E = \varepsilon_0 \chi(\omega) E$$

where $\omega_0 = (K/m)^{1/2}$ is the *effective resonance frequency* of the *bound* electrons, $\chi_0 = Ne^2/(\varepsilon_0 m \omega_0{}^2)$

An optical resonance phenomenon

- The polarization formula is similar to the *amplitude formula for a driven harmonic oscillator*, as it is the displacement of the elastically bound electrons that actually constitutes the polarization.

- We should therefore expect to find an ***optical resonance*** phenomenon occurring for light frequencies in the neighborhood of the resonance frequency ω_0.

- We will see that this resonance phenomenon is manifest as a *large change in the index of refraction* of the medium and also by a *strong absorption* of light at or near the resonance frequency.

Frequency-dependent susceptibility

- Thus we have the *frequency-dependent susceptibility*

$$\chi(\omega) = \chi_0 \frac{\omega_0^{\;2}}{\omega_0^{\;2} - \omega^2 - i\omega\Gamma}$$

- The *real* and *imaginary* parts of $\chi(\omega)$, denoted $\chi'(\omega)$ and $\chi''(\omega)$,

$$\chi'(\omega) = \chi_0 \frac{\omega_0^{\;2}(\omega_0^{\;2} - \omega^2)}{(\omega_0^{\;2} - \omega^2)^2 + (\omega\Gamma)^2}$$

$$\chi''(\omega) = \chi_0 \frac{\omega_0^{\;2}\omega\Gamma}{(\omega_0^{\;2} - \omega^2)^2 + (\omega\Gamma)^2}$$

Susceptibility near resonance

- The behavior of $\chi(\omega)$ in the vicinity of resonance ($\omega \sim \omega_0$) is often of particular interest.
- In this region, we may use the approximation $(\omega_0^2 - \omega^2) = (\omega_0+\omega)(\omega_0-\omega) \approx 2\omega_0\,(\omega_0 - \omega)$

$$\chi(\omega \approx \omega_0) \approx \chi_0 \frac{\omega_0 / 2}{(\omega_0 - \omega) - i\Gamma / 2} = \frac{-\chi_0}{2} \frac{\omega_0}{(\omega - \omega_0) + i\Gamma / 2}$$

$$\chi''(\omega \approx \omega_0) \approx \frac{\chi_0}{2} \frac{\omega_0 \Gamma / 2}{(\omega - \omega_0)^2 + (\Gamma / 2)^2}$$

$$\chi'(\omega \approx \omega_0) \approx \frac{-\chi_0}{2} \frac{\omega_0 (\omega - \omega_0)}{(\omega - \omega_0)^2 + (\Gamma / 2)^2}$$

Γ = FWHM of χ'', a Lorentzian resonance

Susceptibility far from resonance

- The behavior of $\chi(\omega)$ *far from* resonance is also of interest.
- In the limit $|(\omega - \omega_0)| >> \Gamma$, the susceptibility is approximately *real*

$$\chi(\omega) \approx \chi_0 \frac{{\omega_0}^2}{{\omega_0}^2 - \omega^2}$$

s.t. that the medium exhibits *negligible absorption*.

- The *absorption coefficient* and the *refractive index* of a resonant medium may be determined from the expressions for $\chi'(\omega)$ and $\chi''(\omega)$.

Polarization effect on the propagation

- In order to show how the polarization affects the propagation of light, we return to the *general wave equation*.
- For a dielectric there is *no* conduction term.
- Hence we have

$$\nabla\times(\nabla\times E)+\frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}=\frac{-\mu_0 N e^2}{m}\left(\frac{1}{\omega_0{}^2-\omega^2-i\Gamma\omega}\right)\frac{\partial^2 E}{\partial t^2}$$

- For an *isotropic* material with *no* free charges,

$$\nabla\bullet E=0,\nabla\times(\nabla\times E)=-\nabla^2 E$$

$$\nabla^2 E=\frac{1}{c^2}\left(1+\frac{Ne^2}{m\varepsilon_0}\frac{1}{\omega_0{}^2-\omega^2-i\Gamma\omega}\right)\frac{\partial^2 E}{\partial t^2}$$

Solving the general wave equation

- Try a solution of the *plane harmonic wave* form

$$E = E_0 \exp i(kz - \omega t)$$

- Direct substitution shows that this is a possible solution provided that

$$k^2 = \frac{\omega^2}{c^2}\left(1 + \frac{Ne^2}{m\varepsilon_0}\frac{1}{\omega_0{}^2 - \omega^2 - i\Gamma\omega}\right) = \frac{\omega^2}{c^2}n^2$$

- The presence of the imaginary term in the denominator implies that the *wavenumber* k (and the *refractive index* n) must be a *complex* number.

Complex refractive index

- We have the complex index n

$$n^2 = (n' + i n'')^2 = 1 + \frac{Ne^2}{m\varepsilon_0} \frac{1}{\omega_0{}^2 - \omega^2 - i\Gamma\omega}$$

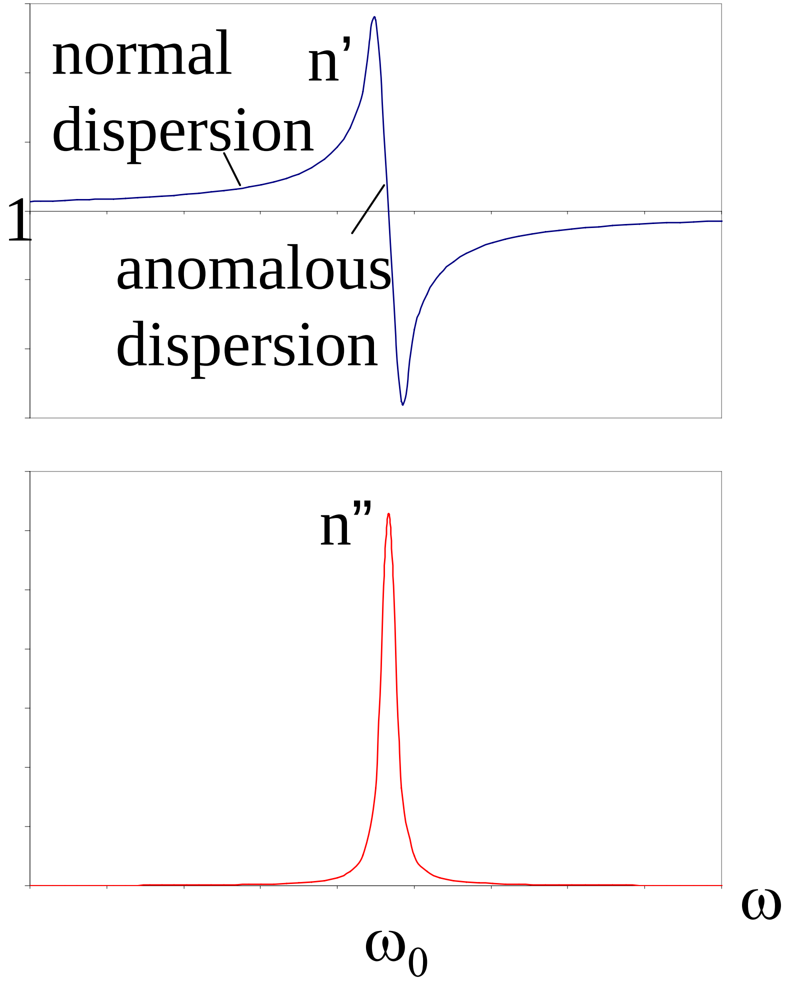
- Equating real and imaginary parts

$$n'^2 - n''^2 = 1 + \frac{Ne^2}{m\varepsilon_0} \frac{\omega_0{}^2 - \omega^2}{(\omega_0{}^2 - \omega^2)^2 + \Gamma^2\omega^2}$$

$$2n' n'' = \frac{Ne^2}{m\varepsilon_0} \frac{\Gamma\omega}{(\omega_0{}^2 - \omega^2)^2 + \Gamma^2\omega^2}$$

Index of refraction and extinction coefficient vs. frequency near a single resonance

- The absorption is strongest at the resonance frequency ω_0.
- The index of refraction is *greater* than unity for small frequencies and increases with frequency as the resonance frequency is approached. This is the case of **“normal” dispersion**, which is exhibited by most transparent substances over the visible region of the spectrum, the principal resonance frequencies being in the *ultraviolet* region.
- At or near the resonance frequency, however, the dispersion becomes **“anomalous”** in the sense that the index of refraction decreases with increasing frequency.



Media with multiple resonances

- Now, so far we have assumed that *all* of the electrons were identically bound, and hence all had the same resonance frequencies.
- A typical dielectric medium contains *multiple* resonances corresponding to different *lattice* and *electronic* vibrations.
- In order to take into account the fact that different electrons may be bound differently, we may assume that a certain fraction f_1 has an associated resonance frequency ω_1, a fraction f_2 has the resonance frequency ω_2, and so on.
- The resulting formula for the square of the complex index of refraction is

oscillator strengths

$$n^2 = 1 + \frac{Ne^2}{m\varepsilon_0} \sum_j \frac{f_j}{\omega_j^{\ 2} - \omega^2 - i\Gamma_j \omega}$$

damping constants

Sellmeier equation

- If the damping constants Γ_j are sufficiently small ($|\omega_j-\omega| >> \Gamma_j$) so that the terms $\Gamma_j\omega$ can be neglected in comparison to the quantities $\omega_j^2-\omega^2$, then the index of refraction is essentially real and its square is given by

$$n^2 \approx 1+\frac{Ne^2}{m\varepsilon_0}\sum_j \frac{f_j}{\omega_j^2-\omega^2}$$

- When expressed in terms of wavelength instead of frequency, the equation is known as ***Sellmeier equation***.
- The Sellmeier equation provides a good description of the refractive index for most optically transparent materials.

Sellmeier equations for selected materials at room temperature

Material	Sellmeier Equation (wavelength λ in μm)	Wavelength range (μm)
Fused silica	$n^2 = 1 + \frac{0.6962\lambda^2}{\lambda^2 - (0.06840)^2} + \frac{0.4079\lambda^2}{\lambda^2 - (0.1162)^2} + \frac{0.8975\lambda^2}{\lambda^2 - (9.8962)^2}$	0.21-3.71
Si	$n^2 = 1 + \frac{10.6684\lambda^2}{\lambda^2 - (0.3015)^2} + \frac{0.0030\lambda^2}{\lambda^2 - (1.1347)^2} + \frac{1.5413\lambda^2}{\lambda^2 - (1104.0)^2}$	1.36-11
GaAs	$n^2 = 3.5 + \frac{7.4969\lambda^2}{\lambda^2 - (0.4082)^2} + \frac{1.9347\lambda^2}{\lambda^2 - (37.17)^2}$	1.4-11
BBO	$n_o^2 = 2.7359 + \frac{0.01878}{\lambda^2 - 0.01822} - 0.01354\lambda^2$	0.22-1.06
	$n_e^2 = 2.3753 + \frac{0.01224}{\lambda^2 - 0.01667} - 0.01516\lambda^2$	0.22-1.06
KDP	$n_o^2 = 1 + \frac{1.2566\lambda^2}{\lambda^2 - (0.09191)^2} + \frac{33.8991\lambda^2}{\lambda^2 - (33.3752)^2}$	0.4-1.06
	$n_e^2 = 1 + \frac{1.1311\lambda^2}{\lambda^2 - (0.09026)^2} + \frac{5.7568\lambda^2}{\lambda^2 - (28.4913)^2}$	0.4-1.06
$LiNbO_3$	$n_o^2 = 2.3920 + \frac{2.5112\lambda^2}{\lambda^2 - (0.217)^2} + \frac{7.1333\lambda^2}{\lambda^2 - (16.502)^2}$	0.4-3.1
	$n_e^2 = 2.3247 + \frac{2.2565\lambda^2}{\lambda^2 - (0.210)^2} + \frac{14.503\lambda^2}{\lambda^2 - (25.915)^2}$	0.4-3.1

Signal velocity in a dispersive medium

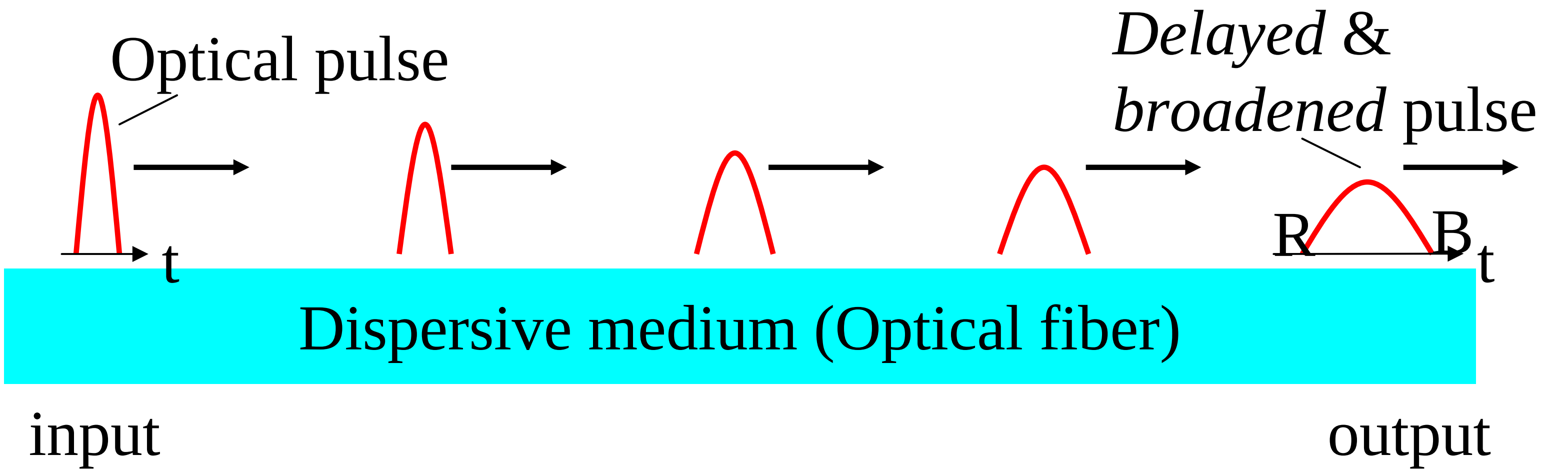
Group velocity as signal velocity

- We discussed in Lecture 6 that *practically* the ***group velocity*** of light corresponds to the velocity of signals.
- Recall

$$v_g = \frac{c_0}{n_g} = \frac{c_0}{n - \lambda \frac{dn}{d\lambda}}$$

- This is fine when the light wavelength is *far away from a resonance*, where we see from the example of fused silica glass that the medium in this spectral range is transparent and in the *normal dispersion* region ($dn/d\lambda < 0$), meaning $n_g > n$, consequently $v_g < c/n$.
- In this range, v_g measures the velocity of a wave packet which can represent a signal. *Signal transmission applications almost always use this spectral range*.

Group velocity as signal velocity



- Recall that v_g itself depends on λ, known as *group velocity dispersion* (GVD), given by

$$D_\lambda = -\frac{\lambda}{c_0}\frac{d^2 n}{d\lambda^2}$$

- Thus, even if $dn/d\lambda < 0$, D_λ can be > 0 (*anomalous GVD*) or < 0 (*normal GVD*) depending on the spectral range. The illustration above corresponds to $D_\lambda < 0$ (Red travels faster than blue).

What about within an absorption band?

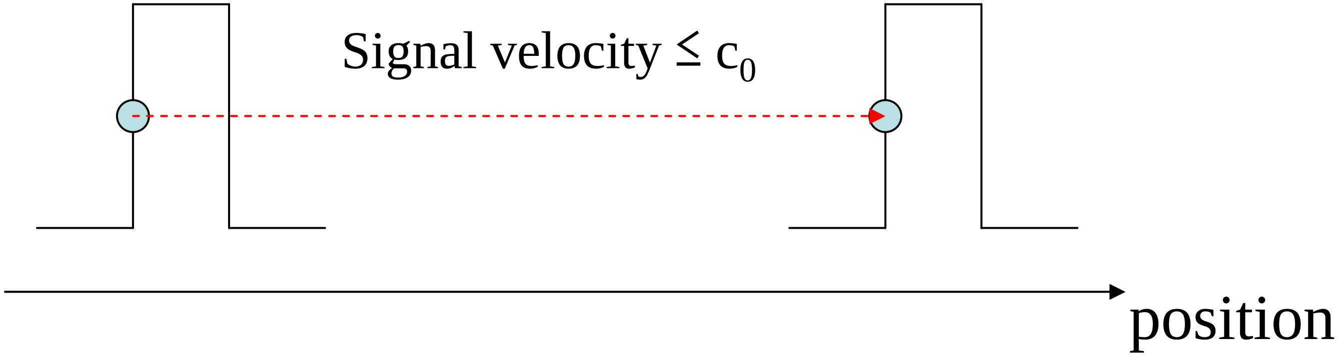
- ***Within an absorption band***, the refractive index n and the group index n_g undergo rapid changes near the resonance frequency.

- We encounter within an absorption band ***anomalous dispersion***, meaning $dn/d\lambda > 0$.
- Thus, it is possible that $n_g < 1$ or even significantly < 0 !
- Consequently, v_g may be *greater than* the velocity of light in free space, c_o, or even be *negative* !

- This raises the question of a potential conflict with *causality* and the *special theory of relativity*, which provides that ***signals (or information) cannot be transmitted at a velocity greater than c_0***.

So what is signal velocity, really?

- It turns out that there is no such conflict as *neither the group velocity* $\mathbf{v}_g$ *nor the phase velocity* $\mathbf{v}_\mathrm{p}$ *corresponds to the* ***signal velocity***, which is ***the speed at which signal (or information) is transmitted between two points***.
- The signal velocity may be determined by tracing the propagation of the onset of a rectangular pulse. It cannot exceed c_0.



Ref. Wave propagation and group velocity, L. Brillouin, Academic Press, 1960